DIAMOND, SCALES AND GCH DOWN TO $\aleph_{\omega^2}$

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Abstract. Gitik and Rinot [3] proved assuming the existence of a supercompact that it is consistent to have a strong limit cardinal κ of countable cofinality such that $2^\kappa = \kappa^+$, there is a very good scale at κ, and $\diamond$ fails along some reflecting stationary subset of $\kappa^+ \cap \mathrm{cof}(\omega)$. In this paper, we force over Gitik and Rinot's model but with a modification of Gitik-Sharon [4] diagonal Prikry forcing to get this result for $\kappa = \aleph_{\omega^2}$.

1. Introduction

In this paper, $\mathrm{cf}(\alpha)$ is the cofinality of α, $\mathrm{cof}(\alpha)$ is the class of ordinals with cofinality α and $\mathrm{cof}(\neq \alpha)$ is the class of ordinals with cofinality not α.

Let VGS_κ denote the presence of a very good scale at κ, BS_κ the presence of a bad scale at κ, $\square_\kappa$ the presence of a square sequence at κ, $\square^*_\kappa$ the presence of a weak square sequence at κ, AP_κ the Approachability Property at κ and $SAP\kappa$ the Stationary Approachability Property at κ. SAP_κ was defined by Rinot [6] while the rest were defined by Cummings, Foreman and Magidor [2]. Recall that for a stationary set $S \subset \kappa^+$, $\diamond_S$ is the assertion that there exists a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that for every $A \subset \kappa^+$, $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary. We will also write GCH_κ to mean that $2^\kappa = \kappa^+$ and SCH_κ to mean if κ is strong limit, then $2^\kappa = \kappa^+$.

Shelah [8] showed that for uncountable λ, $GCH_\lambda \Rightarrow \diamond_S$ for any stationary $S \subset \lambda^+ \cap \mathrm{cof}(\neq \mathrm{cf}(\lambda))$. Shelah [7] showed it is consistent to have GCH_λ but $\neg\diamond_S$ with $S \subset \lambda^+ \cap \mathrm{cof}(\mathrm{cf}(\lambda))$ a non-reflecting stationary set. Gitik and Rinot [3] found a model for $GCH_\kappa + VGS_\kappa + \neg\diamond_S$, where κ is strong limit and $S \subset \kappa^+ \cap \mathrm{cof}(\omega)$ reflects stationarily often. They started with a model with GCH and κ supercompact, performed an Easton support iteration to add a stationary set $S_\alpha \subset \alpha^{+\omega+1}$ and kill all possible diamond sequences along $S_\alpha^{+\omega+1}$ for each inaccessible $\alpha \leqslant \kappa$, performed another Easton support iteration to make $2^\alpha = \alpha^{+\omega+1}$ for each inaccessible $\alpha \leqslant \kappa$, and lastly forced with a supercompact Prikry poset to get GCH_κ and singularize κ^{+n} for $0 \leqslant n < \omega$ to have countable cofinality. Zeman [10], building on work by Shelah [8], showed that when κ is singular, $GCH_\kappa + \neg\diamond_S$ for $S \subset \kappa^+ \cap \mathrm{cof}(\mathrm{cf}(\kappa))$ stationary and reflecting stationarily often implies $\neg\square^*_\kappa$. Below we summarize relevant results.

For every singular cardinal κ

(1) (Shelah) $\square_\kappa \Rightarrow \square^*_\kappa \Rightarrow AP_\kappa \Rightarrow \neg BS_\kappa$
(2) (Cummings-Foreman-Magidor) $\square_\kappa \Rightarrow VGS_\kappa$, but $\square^*_{\aleph_\omega} \not\Rightarrow VGS_{\aleph_\omega}$
(3) (Rinot) $\square^*_\kappa \Rightarrow SAP_\kappa$ but $SAP_{\aleph_\omega} \not\Rightarrow \square^*_{\aleph_\omega}$
(4) (Gitik-Sharon) $VGS_{\aleph_{\omega^2}} \not\Rightarrow AP_{\aleph_{\omega^2}}$
(5) (Rinot) $2^\kappa = \kappa^+ \wedge SAP_\kappa \Rightarrow \diamond_S$ along every reflecting stationary $S \subset \kappa^+$
(6) (Gitik-Rinot) $GCH \wedge AP_{\aleph_\omega} \not\Rightarrow \diamond_S$ along every reflecting stationary set $S \subset \aleph_{\omega+1}$
(7) (Gitik-Rinot) $2^\kappa = \kappa^+ \wedge VGS_\kappa \not\Rightarrow \diamond_S$ along every reflecting stationary set $S \subset \kappa^+$

In this paper, motivated by [1], [3] and [4], we obtain the following:

Theorem 1. *Assuming the consistency of a supercompact, there is a model where $\aleph_{\omega^2}$ is strong limit and $GCH_{\aleph_{\omega^2}} + VGS_{\aleph_{\omega^2}} + \neg \diamond_S + BS_{\aleph_{\omega^2}}$, where $S \subset \aleph_{\omega^2+1} \cap$ cof(ω) reflects stationarily often.*

Both the failure of diamond along S and the prescence of the bad scale imply $\neg\square^*_{\aleph_{\omega^2}}$ in this model.

In the notation of [3,§1], let $Q(\lambda^+) := S(\lambda^+) * KAD(\dot{S}(\lambda^+))$, where $S(\lambda^+)$ is the poset to add a new stationary subset $S \subset \lambda^+ \cap \mathrm{cof}(\omega)$ and $KAD(\dot{S}(\lambda^+))$ is an iterated forcing to enumerate all possible diamond sequences on S and force diamond to fail on all these sequences.

Our construction will proceed as follows. It is broadly similar to the construction in Theorem 1.11 of [3], except we must modify the diagonal Prikry forcing in the final step.

- Start with $V_0 \models$ GCH, κ supercompact.
- Step 1a: Force with $\langle Q(\alpha^{+\omega+1}) : \alpha \leqslant \kappa, \alpha \text{ inaccessible}\rangle$ with Easton support. For each $\alpha \leqslant \kappa$ this adds a stationary $S(\alpha) \subset \alpha^{+\omega+1} \cap \mathrm{cof}(\omega)$ such that $\neg\diamond_{S_\kappa}$, while preserving supercompactness of κ. Let $S = S(\kappa)$. Call the resulting model V_1.
- Step 1b Perform Laver preparation as in [5] to make κ indestructibly supercompact with respect to any further κ-directed closed forcing. Call the resulting model V_2^-.
- Step 2: Force with $\mathrm{Add}(\kappa, \kappa^{+\omega+1})$. This makes $2^\kappa = \kappa^{+\omega+1}$ and S remains stationary because of the $\kappa^{+\omega+1}$-chain condition. Since κ remains supercompact, S reflects stationarily often. Call the new model V_2.

- Step 3: Force with a modified version of the diagonal Prikry poset with interleaved collapses in [4]. This makes κ into $\aleph_{\omega^2}$, collapses every κ^{+n} to $\aleph_{\omega^2}$ and makes $2^{\aleph_{\omega^2}} = \aleph_{\omega^2+1}$ while preserving stationarity of S and $\neg\diamond_S$.

Step 1a is already described in [3]. In V_2^-, the powerset function behaves wildly below κ, but later we will use measures which concentrate on inaccessible $\alpha < \kappa$ such that $2^\alpha = \alpha^{+\omega+1}$. In Section 2, we perform Step 2 to obtain V_2, our main ground model for the rest of the paper. In Section 3, we carry out Step 3 to obtain our final model and show that S remains a stationary set that reflects stationarily often. The reason we need to modify the diagonal Prikry poset here, unlike in [3], is that in [4] the ground model is prepared to have $2^\kappa = \kappa^{+\omega+2}$, which allows guiding generics for the collapses to be constructed that make conditions with the same stem compatible. Here, guiding generics need not exist, so we must modify the poset to work without them. In Section 4, we show that the final model has a very good scale and a bad scale. In Section 5, we show that $\diamond_S$ continues to fail in the final model.

2. Preparing the Ground Model

Working in V_1, take j_1 to be a $\kappa^{+\omega+1}$-supercompact embedding with critical point κ and j_2 to be a $\kappa^{+\omega+2}$-supercompact embedding with critical point κ. Regard $\mathrm{Add}(\kappa, \kappa^{+\omega+1})$ as consisting of partial functions $p : \kappa^{+\omega+1} \times \kappa \to \kappa$ with $|p| < \kappa$. Then a generic $\hat{G}$ for $\mathrm{Add}(\kappa, \kappa^{+\omega+1})$ will add $\kappa^{+\omega+1}$-many generic functions $F_\alpha : \kappa \to \kappa$, which we may index so that either $\alpha < \kappa^{+\omega+1}$ or $\alpha \leqslant \kappa^{+\omega+1}$.

Lemma 2. *There are lifts of j_1 to j_1^* and j_2 to j_2^* both in $V_2 := V_2^-[\hat{G}]$ so that for every $\alpha < \kappa^{+\omega+1}$ there is $f_\alpha : \kappa \to \kappa$ such that $j_1^* f_\alpha(\kappa) = \alpha$ and for every $\alpha \leqslant \kappa^{+\omega+1}$ there is $f_\alpha : \kappa \to \kappa$ such that $j_2^* f_\alpha(\kappa) = \alpha$.*

Proof. We will prove both claims simultaneously. Let j denote either j_1 or j_2 and $\delta = \kappa^{+\omega+1}$ if $j = j_1$ and $\delta = \kappa^{+\omega+1} + 1$ if $j = j_2$. By standard arguments, let K' be a generic for $\mathrm{Add}(j(\kappa), j(\kappa)^{+\omega+1})$ over V_2^- such that $j"\hat{G} \subset K'$, $F_\alpha^* : j(\kappa) \to j(\kappa)$ be the generic functions added by K' and assume we have lifted j to $V_2^-[K']$ so that $j(F_\alpha) = F^*_{j(\alpha)}$.

Lemma 3. *There exists K generic for Add$(j(\kappa), j(\kappa)^{+\omega+1})$ over V_2^- such that:*

- *$K \supset j"\hat{G}$*
- *If $\xi < \kappa$, $\alpha < \delta$ and $\langle j(\alpha), \xi\rangle \in$ dom(p) for some $p \in K$, then there is $p' \in K'$ such that $\langle j(\alpha), \xi\rangle \in$ dom(p') and $p(\langle j(\alpha), \xi\rangle) = p'(\langle j(\alpha), \xi\rangle)$*
- *Whenever $\langle j(\alpha), \kappa\rangle \in$ dom(p) and $p \in K$, $p(\langle j(\alpha), \kappa\rangle) = \alpha$*

Proof. Define $p \in K \iff$

- If $\langle j(\alpha), \kappa\rangle \in \mathrm{dom}(p)$, then $p(\langle j(\alpha), \kappa\rangle) = \alpha$
- $p|(\mathrm{dom}(p)\backslash\{\langle j(\alpha), \kappa\rangle : \alpha < \delta\}) \in K'$.

Since $|p| < \kappa$ for every $p \in \hat{G}$, $j(p) = j"p$. But $\kappa \notin \mathrm{im}(j)$. So $\langle j(\alpha), \kappa\rangle \notin \mathrm{dom}(j(p))$, which implies $j"\hat{G} \cap K = j"\hat{G} \cap K'$. Since $j"\hat{G} \subset K'$, we have $j"\hat{G} \subset K$. To show K is generic, let $A \subset \mathrm{Add}(j(\kappa), j(\kappa^{+\omega+1}))$ be a maximal antichain. Let $q = \{\langle j(\alpha), \kappa\rangle \mapsto \alpha : \alpha < \delta\}$. Then $|q| = \kappa^{+\omega+1}$ and hence $q \in \mathrm{Add}(j(\kappa), j(\kappa)^{+\omega+1})$. For each $p \in \mathrm{Add}(j(\kappa), j(\kappa)^{+\omega+1})$, let $\mathrm{ch}(p) = (p|\mathrm{dom}(p)\backslash\mathrm{dom}(q)) \cup (q|\mathrm{dom}(p) \cap \mathrm{dom}(q))$.

Let $A' = \{p \in \mathrm{Add}(j(\kappa), j(\kappa)^{+\omega+1}) : \mathrm{ch}(p) \in A\}$. Suppose $p_1, p_2 \in A'$ are distinct. If $\mathrm{ch}(p_1) = \mathrm{ch}(p_2)$, then p_1, p_2 are the same outside $\mathrm{dom}(q)$ and have the same domain on $\mathrm{dom}(q)$. So $p_1(\langle j(\alpha), \kappa\rangle) \neq p_2(\langle j(\alpha), \kappa\rangle)$ for some α, which implies $p_1 \perp p_2$. If $\mathrm{ch}(p_1) \neq \mathrm{ch}(p_2)$, then since A is an antichain, $\mathrm{ch}(p_1) \perp \mathrm{ch}(p_2)$. This incompatibility must be witnessed by some input outside $\mathrm{dom}(q)$, which will also witness that $p_1 \perp p_2$. Therefore, A' is an antichain.

Now suppose $p' \in \mathrm{Add}(j(\kappa), j(\kappa)^{+\omega+1})$ is incompatible with every element of A'. Define $p = (p'|\mathrm{dom}(p')\backslash\mathrm{dom}(q)) \cup q$. Let $r \in A$ be compatible with p. Then $r(\langle j(\alpha), \kappa\rangle) = \alpha$ whenever $\langle j(\alpha), \kappa\rangle \in \mathrm{dom}(r)$. Define r' with $\mathrm{dom}(r') = \mathrm{dom}(r)$ by $r'(\langle j(\alpha), \kappa\rangle) = p'(\langle j(\alpha), \kappa\rangle)$ whenever $\langle j(\alpha), \kappa\rangle \in \mathrm{dom}(r)$ and $r'|\mathrm{dom}(r)\backslash\mathrm{dom}(q) = r|\mathrm{dom}(r)\backslash\mathrm{dom}(q)$. Then $\mathrm{ch}(r') = r \in A$, so $r' \in A'$. Furthermore, $r' \not\perp p'$ because $r \not\perp p$, r' is compatible with r outside $\mathrm{dom}(q)$ and p' is compatible with p outside $\mathrm{dom}(q)$. But this is a contradiction because p' is incompatible with every element of A'. Therefore, p is incompatible with every element of A, which contradicts maximality of A. It follows that A' must be a maximal antichain.

Since K' is generic, let $p \in A' \cap K'$. Then $\mathrm{ch}(p) \in A \cap K$. It follows that K is generic. □

Let j^* be such that $j^*(F_\alpha)(\xi) = p(\langle j(\alpha), \xi\rangle)$ for any $p \in K$ with $\langle j(\alpha), \xi\rangle \in \mathrm{dom}(p)$. This completes the proof of Lemma 2. □

3. The Main Forcing

Let J_1, J_2 be given by Lemma 2 and $U, \bar{U}$ their corresponding ultrafilters on $P_\kappa(\kappa^{+\omega+1})$ and $P_\kappa(\kappa^{+\omega+2})$ respectively. From now on, we will write $j_{\bar{U}}$ in place of J_2 and j in place of J_1. Also, let U_n be the projection of $\bar{U}$ on to $P_\kappa(\kappa^{+n})$ with corresponding elementary embedding $j_n : V_2 \to M_n$. For convenience, we will write κ_x for $\kappa \cap x$ and when $x \in P_\kappa(\kappa^{+i}), y \in P_\kappa(\kappa^{+j})$ with $i < j$, then $x \prec y$ means $x \subset y$ and $ot(x) < \kappa_y$.

We define in V_2 another forcing poset $\mathbb{P}$ that will collapse every κ^{+n} to κ and make κ into $\aleph_{\omega^2}$. Conditions will be of the form
$\langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}, A_n, C_n, A_{n+1}, C_{n+1}, ...\rangle$, where

- $d \in \mathrm{Col}(\omega_1, < \kappa_{x_0})$ if $n > 0$ and $d \in \mathrm{Col}(\omega_1, < \kappa)$ if $n = 0$
- $x_i \in P_\kappa(\kappa^{+i})$, $x_i \prec x_{i+1}$
- $c_i \in \mathrm{Col}(\kappa_{x_i}^{+\omega+2}, < \kappa_{x_{i+1}})$ for $i < n-1$ and $c_{n-1} \in \mathrm{Col}(\kappa_{x_{n-1}}^{+\omega+2}, < \kappa)$
- $A_i \in U_i$ and $x_{n-1} \prec y$ whenever $y \in A_i$
- C_i is a function with domain A_i and for every $x \in A_i$, $C_i(x) \in \mathrm{Col}(\kappa_x^{+\omega+2}, < \kappa)$, i.e. $[C_i]_{U_i} \in \mathrm{Col}^{M_i}(\kappa^{+\omega+2}, < j_i(\kappa))$

We will also require that each x_i is such that $\kappa \cap x_i$ is inaccessible. Note that this happens on a set of U_i-measure 1.

For any condition $p = \langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}, A_n, C_n, ...\rangle$, denote $\mathrm{stem}(p) = \langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}\rangle$ and $\mathrm{length}(p) = n$.

Given $p = \langle d^p, x_0^p, c_0^p, ..., x_{n-1}^p, c_{n-1}^p, A_n^p, C_n^p, ...\rangle$ and
$q = \langle d^q, x_0^q, c_0^q, ..., x_{m-1}^q, c_{n-1}^q, A_n^q, C_n^q, ...\rangle$, define $p \leqslant q$ if

- $m \leqslant n$
- $d^p \leqslant d^q$
- $x_i^p = x_i^q$ and $c_i^p \leqslant c_i^q$ for $i < m$
- $x_i^p \in A_i^q$ and $c_i^p \leqslant C_i^q(x_i^p)$ for $m \leqslant i < n$
- $A_i^p \subset A_i^q$ and $C_i^p(x) \leqslant C_i^q(x)$ for $x \in A_i^p$ for $i \geqslant n$

Define $p \leqslant^* q$ if $p \leqslant q \wedge \mathrm{length}(p) = \mathrm{length}(q)$.

For stems $h = \langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}\rangle$ and $h' = \langle d', x_0', c_0', ..., x_{n-1}', c_{n-1}'\rangle$ of the same length, define $h \leqslant h'$ if $d \leqslant d'$, $x_i = x_i'$ and $c_i \leqslant c_i'$ for $i < n$. If $p = \langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}, A_n, C_n, ...\rangle$, $x_i \in A_i$ for $n \leqslant i \leqslant n+k$ and $x_i \prec x_{i+1}$ for $n \leqslant i < n+k$, let $p^\frown\langle x_n, ..., x_{n+k}\rangle$ be the weakest extension of p by $\langle x_n, ..., x_{n+k}\rangle$, i.e.
$\langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}, x_n, C_n(x_n), ..., x_{n+k}, C_{n+k}(x_{n+k}), A_{n+k+1}, C_{n+k+1}, ...\rangle$.

$\mathbb{P}$ adds two new generic sequences of interest: A Prikry sequence $\langle x_n : n < \omega\rangle$, and a sequence of collapse generics $\langle c_n : n < \omega\rangle$. Let $\kappa_n = \kappa \cap x_n$. Then $\langle \kappa_n : n < \omega\rangle$ singularizes κ to have countable cofinality while $\langle c_n : n < \omega\rangle$ collapses all cardinals in $(\kappa_n^{+\omega+2}, \kappa_{n+1})$ to $\kappa_n^{+\omega+2}$ for every n.

From now on let G be a generic for $\mathbb{P}$ and $V_3 = V_2[G]$. In V_3, all cardinals from κ to $(\kappa^{+\omega})^{V_2}$ have cofinality ω and κ becomes $\aleph_{\omega^2}$. We will show later that $(\kappa^{+\omega+1})^{V_2}$ is the new successor of κ.

$\mathbb{P}$ satisfies a property characteristic of Prikry type forcings.

Lemma 4. ***Prikry Property*** *For any formula $\varphi(v_1, ..., v_m)$, parameters $a_1, ..., a_m \in V_3$ and any condition $p \in \mathbb{P}$, there is a condition $r \leqslant^* p$ such that $r \Vdash \varphi(\dot{a_1}, ..., \dot{a_m})$ or $r \Vdash \neg\varphi(\dot{a_1}, ..., \dot{a_m})$.*

Proof. For convenience, we write φ instead of $\varphi(\dot{a_1}, ..., \dot{a_m})$.

Claim 5. *Let $k \geqslant 0$. Then for every $r \in \mathbb{P}$, if $n = length(r)$ and $\mathcal{A} = \{\langle x_n, ..., x_{n+k}\rangle : x_i \in A_i^r, x_j \prec x_{j+1}\}$, then there is $r' \leqslant^* r$ such that for all $\vec{x} \in \mathcal{A}$, if $q \leqslant^* r'^\frown\vec{x}$ and $q||\varphi$, then $r'^\frown\vec{x}||\varphi$.*

Proof. First consider the case $k = 0$. Let $\langle x_\alpha : \alpha < \kappa^{+n}\rangle$ enumerate A_n^r. We inductively construct a sequence $\langle q_\alpha : \alpha < \kappa^{+n}\rangle$ as follows: If there is $q \leqslant^* r^\frown x_\alpha$ such that $q||\varphi$, choose one and call it q_α. If not, let $q_\alpha = r^\frown x_\alpha$. During the construction, maintain inductively that for each $i > n$, $\langle [C_i^{q_\alpha}]_{U_i}\rangle$ is decreasing by strengthening the q_α if necessary; we can do this because $\mathrm{Col}^{M_i}(\kappa^{+\omega+2}, < j_i(\kappa))$ is closed under κ^{+i}-sequences. Let $q_x = q_\alpha$, where α is such that $x = x_\alpha$.

We now define $r' \leqslant^* r$ as follows:

- $|\mathrm{Col}(\omega_1, < \kappa_{x_0})|, |\mathrm{Col}(\kappa_{x_i}^{+\omega+2}, < \kappa_{x_{i+1}})| < \kappa$ for $i < n-1$. So there is $A' \in U_n, A' \subset A_n^r$ such that on A', $x \mapsto d^{q_x}$ and $x \mapsto c_i^{q_x}$ for $i < n-1$ are constant. Let $d^{r'}, c_i^{r'}$ be those constants.
- $x \mapsto c_{n-1}^{q_x} \in \mathrm{Col}(\kappa_{x_{n-1}}^{+\omega+2}, < \kappa_x)$, which can be coded as a subset of κ_x. So $c_{n-1}^{q_x}$ can be coded as a bounded subset of κ_x. By Fodor's Lemma, there is $A'' \in U_n, A'' \subset A_n^r$ such that $x \mapsto c_{n-1}^{q_x}$ is constant. Let $c_{n-1}^{r'}$ be that constant.
- Let $A_n^{r'} = A' \cap A''$
- For $x \in A_n^{r'}$, let $C_n^{r'}(x) = c_n^{q_x}$
- For $i > n$, we can find a lower bound $[b_i]_{U_i} \in \mathrm{Col}^{M_i}(\kappa^{+\omega+2}, < j_i(\kappa))$ for $\langle [C_i^{q_x}]_{U_i} : x \in A_n\rangle$, where b_i has the full domain $P_\kappa(\kappa^{+i})$. Let $B_i^x = \{y \in P_\kappa(\kappa^{+i}) : b_i(y) \leqslant C_i^{q_x}(y)\}$ and $B_i = \Delta_{x\in A_n} B_i^x$. Also let $B_i' = \Delta_{x\in A_n} A_i^{q_x}$. Take $A_i^{r'} = A_i^r \cap A' \cap A'' \cap B_i \cap B_i'$.
- For $i > n$, let $C_i^{r'} = b_i | A_i^{r'}$.

For all $x \in A_n^{r'}$, $r'^\frown x \leqslant^* q_x$. So r' is as desired.

Now assume the claim holds for some k. Let $\mathcal{A} = \{\langle x_n, ..., x_{n+k+1}\rangle : x_i \in A_i^r, x_j \prec x_{j+1}\}$ and for each $x \in A_n^r$, $\mathcal{A}_x = \{\langle x_{n+1}, ..., x_{n+k+1}\rangle : \langle x, x_{n+1}, ..., x_{n+k+1}\rangle \in \mathcal{A}\}$. Apply the induction hypothesis to each $r^\frown x$ and $\mathcal{A}_x$ to obtain $q_x \leqslant^* r^\frown x$ such that for all $\vec{x} \in \mathcal{A}_x$, if $q \leqslant^* q_x{}^\frown\vec{x}$ and $q||\varphi$, then $q_x{}^\frown\vec{x}||\varphi$. As before, do this inductively, maintaining that $\langle [C_i^{q_x}]_{U_i}\rangle$ is decreasing (with respect to some well-ordering of A_n^r) for $i > n$. We then use the same argument as in the $k = 0$ case to find $r' \leqslant^* r$ such that for any $x \in A_n^{r'}$, $r'^\frown x \leqslant^* q_x$. If now $x^\frown\vec{x} \in \mathcal{A}$ and $q \leqslant^* r'^\frown x^\frown\vec{x}$ with $q||\varphi$, then $q \leqslant^* q_x^\frown\vec{x}$. So $q_x^\frown\vec{x}||\varphi$ and hence $r'^\frown x^\frown\vec{x}||\varphi$ as desired. □

Let $n = \mathrm{length}(p)$. Using the claim, inductively construct $\langle p_k : k < \omega\rangle$ a $\leqslant^*$-decreasing sequence with $p_0 = p$ such that for all $k \geqslant 1$, if $\vec{x} = \langle x_n, ..., x_{n+k}\rangle \in A_n^{p_k} \times ... \times A_{n+k}^{p_k}$ with $x_i \prec x_{i+1}$ and $q \leqslant^* p_k^\frown\vec{x}$, then $q||\varphi \Rightarrow p_k^\frown\vec{x}||\varphi$. Let r be the weakest lower bound for $\langle p_k : 1 \leqslant k < \omega\rangle$.

Let $Z = \{\langle x_n, ..., x_{n+k}\rangle : (\forall n \leqslant i < n+k) x_i \prec x_{i+1}, (\forall n \leqslant i \leqslant n+k) x_i \in A_i^r\}$

and $F : Z \to 3$ be given by

$$F(\vec{x}) = \begin{cases} 0 & r^\frown\vec{x} \Vdash \varphi \\ 1 & r^\frown\vec{x} \Vdash \neg\varphi \\ 2 & otherwise \end{cases}$$

By standard results, there is $\langle H_i : i \geqslant n\rangle$ with $H_i \in U_i$, $H_i \subset A^r_i$ such that for each $k \geqslant n$, $F|\{\langle x_n, ..., x_{n+k}\rangle \in Z : x_i \in H_i\}$ is constant. Let r' be obtained from r by intersecting it's measure one sets with the H_i, i.e. $\text{stem}(r') = \text{stem}(r)$, $A^{r'}_i = A^r_i \cap H_i$ and $C^{r'}_i = C^r_i|A^r_i \cap H_i$ for $i \geqslant n$. We claim that r' is as desired.

Suppose r' does not decide φ. Then there are $q_0, q_1 \leqslant r'$ such that $q_0 \Vdash \varphi$ and $q_1 \Vdash \neg\varphi$. Without loss of generality, assume $\text{length}(q_0) = \text{length}(q_1) = n+k$, where $k \geqslant 1$. Then $q_0 \leqslant^* p^\frown_k\langle x^{q_0}_n, ..., x^{q_0}_{n+k}\rangle$. By the claim just proven, $p^\frown_k\langle x^{q_0}_n, ..., x^{q_0}_{n+k}\rangle \Vdash \varphi$. By the same argument, $p^\frown_k\langle x^{q_1}_n, ..., x^{q_1}_{n+k}\rangle \Vdash \neg\varphi$. But this contradicts $F|\{\vec{x} : |\vec{x}| = k\}$ being constant. So $r'||\varphi$. □

A similar argument establishes the following strengthening of the Prikry Property. We omit the proof, but prove a useful corollary.

Lemma 6. *If $D \subset \mathbb{P}$ is dense and $p \in \mathbb{P}$, there is $q \leqslant^* p$ and $n \geqslant$ length(p) such that whenever $r \leqslant q$ and length$(r) = n$, $r \in D$.*

Corollary 7. *$(\kappa^{+\omega+1})^{V_2}$ remains a regular cardinal after forcing with $\mathbb{P}$.*

Proof. $\mathbb{P}$ collapses $(\kappa^{+\omega})^{V_2}$ to κ and singularizes κ, so it is enough to show that there is no unbounded $h : \tau \to (\kappa^{+\omega+1})^{V_2}$ in V_3 with $\tau < \kappa$.

Suppose not and let $p \Vdash \dot{h} \to (\kappa^{+\omega+1})^{V_2}$ unbounded. For each $\alpha < \tau$, let $D_\alpha = \{q \in \mathbb{P} : (\exists\beta)q \Vdash \dot{h}(\alpha) = \beta\}$ and note these are dense and downwards closed. For convenience, let $p_{-1} = p$. Inductively construct a $\leqslant^*$-decreasing sequence $\langle p_\alpha : \alpha < \tau\rangle$ and $\langle n_\alpha : n < \tau\rangle$ as follows:

- Given p_α, apply Lemma 6 to p_α and D_α to obtain $p_{\alpha+1} \leqslant^* p_\alpha$ and $n_{\alpha+1}$ as in the conclusion.
- If α is a limit ordinal, given p_β for all $\beta < \alpha$, let p' be a $\leqslant^*$-lower bound for $\langle p_\beta : \beta < \alpha\rangle$. Apply Lemma 6 to p' and D_α to obtain p_α and n_α as in the conclusion.

Let $q \leqslant^* p_\alpha$ for all $\alpha < \tau$. Then for any $r \leqslant q$, if $\text{length}(r) \geqslant n_\alpha$, $r \Vdash \dot{h}(\alpha) = \beta_\alpha$ for some β_α. Fix an arbitrary $\leqslant$-decreasing sequence $\langle r_n : n < \omega\rangle \in V_2$ below q with $\text{length}(r_n) \geqslant n$. Let $f(\alpha) = \beta \iff (\exists n)r_n \Vdash \dot{h}(\alpha) = \beta$. Then $f \in V_2$ is a well-defined unbounded function from τ to $\kappa^{+\omega+1}$. This is a contradiction. □

Lemma 8. *If $\langle Y_n : n < \omega\rangle$ is a sequence of sets in V_2 with $Y_n \in U_n$ for all n, then $x_n \in Y_n$ for all large enough n.*

Proof. Working in V_2, let $D = \{q \in \mathbb{P} : X_n^q \subset Y_n$ for n $\geqslant$ length$(q)\}$. This is dense in $\mathbb{P}$, since any $p \in \mathbb{P}$ can be strengthened by intersecting each X_n^p with Y_n. So we can find $q \in G \cap D$ that will force $x_n \in Y_n$ for all $n \geqslant$ length(q). □

Let $\mathbb{D}^* = \prod_i \mathbb{C}_i$, where $\mathbb{C}_i = [x \mapsto \mathrm{Col}(\kappa_x^{+\omega+2}, < \kappa)]_{U_i}$. This is exactly $\mathrm{Col}^{M_i}(\kappa^{+\omega+2}, < j_{U_i}(\kappa))$. Let $\mathbb{D} = \prod_i \mathbb{C}_i/\mathit{finite}$, i.e. equivalence classes of elements of $\mathbb{D}$ where two elements $\langle [C_0]_{U_0}, [C_1]_{U_1}, ...\rangle$ and $\langle [C_0']_{U_0}, [C_1']_{U_1}, ...\rangle$ are equivalent iff $\exists n \forall i > n$ $[C_i]_{U_i} = [C_i']_{U_i}$. Denote the equivalence class of $\langle [C_0]_{U_0}, [C_1]_{U_1}, ...\rangle$ by $\langle [C_0]_{U_0}, [C_1]_{U_1}, ...\rangle_{fin}$. Without risk of confusion, we may omit a finite initial segment when writing this. Define $\langle [C_0]_{U_0}, [C_1]_{U_1}, ...\rangle_{fin} \leqslant \langle [C_0']_{U_0}, [C_1']_{U_1}, ...\rangle_{fin}$ iff $\exists n \forall i > n$ $[C_i]_{U_i} \leqslant [C_i']_{U_i}$. This is clearly independent of choice of representatives.

Lemma 9. $\mathbb{P}$ *projects to* $\mathbb{D}$

Proof. Let $\pi : \mathbb{P} \to \mathbb{D}$ be $\pi(\langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}, A_n, C_n, ...\rangle) = \langle [C_n]_{U_n}, [C_{n+1}]_{U_{n+1}}, ...\rangle_{fin}$. This is clearly order-preserving. Suppose $p = \langle x_0, d, c_0, ..., x_{n-1}, c_{n-1}, A_n, C_n, ...\rangle$ and $q \leqslant \langle [C_n]_{U_n}, [C_{n+1}]_{U_{n+1}}, ...\rangle_{fin}$. Let $q = \langle [C_n']_{U_n}, [C_{n+1}']_{U_{n+1}}, ...\rangle_{fin}$. Then $\exists m$ $[C_i']_{U_i} \leqslant [C_i]_{U_i}$ for all $i \geqslant m$. Fix such an m and assume without loss of generality $m \geqslant n$. Then we can extend p to a condition p' of length m such that $[C_i^{p'}]_{U_i} = [C_i']_{U_i}$ for $i \geqslant m$, and $\pi(p') = q$. □

Taking projections of G, let R and G_R be the generics for $\mathbb{D}$ and $\mathbb{P}/R$ over V_2 respectively. Stems and direct extensions are defined in $\mathbb{P}/R$ just as in $\mathbb{P}$.

Lemma 10. $\mathbb{D}$ *is* $\kappa^{+\omega} + 1$*-strategically closed*

Proof. Consider a game of length $\kappa^{+\omega}+1$. We will inductively describe a winning strategy for Player I. Let $\alpha_0 < \kappa^{+\omega}$ be an even ordinal and assume inductively that $\forall n \forall i \geqslant n(\alpha < \kappa^{+n} \Rightarrow \langle [C_i^\beta]_{U_i} : \beta \leqslant \alpha, \beta \text{ even}\rangle$ is decreasing) for every even $\alpha < \alpha_0$. Let v be the least integer such that $\alpha_0 < \kappa^{+v}$.

If α_0 is a successor, this means $\langle [C_i^\beta]_{U_i} : \beta \leqslant \alpha_0 - 2, \beta \text{ even}\rangle$ is decreasing for all $i \geqslant v$. Suppose that Player II plays $\langle [C_0^{\alpha_0-1}]_{U_0}, [C_1^{\alpha_0-1}]_{U_1}, ...\rangle_{fin}$ at stage $\alpha_0 - 1$. Then $[C_i^{\alpha_0-1}]_{U_i} \leqslant [C_i^{\alpha_0-2}]_{U_i}$ for all $i \geqslant w$ for some w. Without loss of generality, assume $w \geqslant v$. Then letting Player I play $\langle [C_0^{\alpha_0}]_{U_0}, [C_1^{\alpha_0}]_{U_1}, ...\rangle_{fin}$, where $[C_i^{\alpha_0}]_{U_i} = [C_i^{\alpha_0-2}]_{U_i}$ for $v \leqslant i < w$ and $[C_i^{\alpha_0}]_{U_i} = [C_i^{\alpha_0-1}]_{U_i}$ for $i \geqslant w$ continues the game while maintaining the induction hypothesis. Each $\mathbb{C}_i$ is κ^{+i}-closed. So if $\alpha < \kappa^{+n}$ and $i > n$, we may take $[C_i^\alpha]_{U_i}$ to be a lower bound for $\langle [C_i^\beta]_{U_i} : \beta < \alpha\rangle$. This maintains the induction hypothesis.

If α_0 is a limit ordinal, this means $\langle [C_i^\beta]_{U_i} : \beta \leqslant \alpha_0, \beta \text{ even}\rangle$ is decreasing for $i \geqslant v$. Since $\mathbb{C}_i$ is κ^{+i}-closed, we can find $[C_i^{\alpha_0}]$ a lower bound for $\langle [C_i^\beta]_{U_i} : \beta \leqslant \alpha_0, \beta \text{ even}\rangle$. Let Player I play $\langle [C_0^{\alpha_0}]_{U_0}, [C_1^{\alpha_0}]_{U_1}, ...\rangle_{fin}$ at stage α_0. This condition is below the move at every earlier even stage, and since α_0 is limit, it is below the move at every earlier odd stage too. So it continues the game while maintaining the induction hypothesis.

At the final stage $\kappa^{+\omega}$, let Player I play $\langle [C_0^{\kappa^{+\omega}}]_{U_0}, [C_1^{\kappa^{+\omega}}]_{U_1}, ...\rangle_{fin}$, where $[C_i^{\kappa^{+\omega}}]_{U_i}$ is a lower bound for $\langle [C_i^\alpha]_{U_i} : \alpha < \kappa^{+i}\rangle$. This is a lower bound for $\langle C^\alpha : \alpha < \kappa^{+\omega}\rangle$, where $C^\alpha = \langle [C_0^\alpha]_{U_0}, [C_1^\alpha]_{U_1}, ...\rangle_{fin}$, because for any $\alpha_0 < \kappa^{+\omega}$, if u is the least integer such that $\alpha_0 < \kappa^{+u}$, then $[C_i^{\kappa^{+\omega}}]_{U_i} \leqslant [C_i^{\alpha_0}]_{U_i}$ for $i \geqslant u$. □

Lemma 11. $\mathbb{D}^*$ *preserves all* V_2*-cardinals* $\leqslant \kappa^{+\omega+1}$.

Proof. $\mathbb{D}^*$ preserves all cardinals $\leqslant \kappa$ by closure. Let us first show that $\mathbb{D}^*$ preserves κ^{+n} for every n. It suffices to show that each $\mathbb{C}_i$ preserves κ^{+n}. This is obvious when $i \geqslant n$ because $\mathbb{C}_i$ is κ^{+i}-closed, so assume $i < n$. Note that $M_i \models \mathbb{C}_i$ is $\kappa^{+\omega+1}$-closed, and hence $\mathbb{C}_i$ preserves $(\kappa^{+n})^{M_i}$. So if $M_i^{\mathbb{C}_i}$ collapses κ^{+n}, then $(\kappa^{+n})^{M_i} \neq \kappa^{+n}$. Now $j_n = k \circ j_i$, where $k : M_i \to M_n$ is given by $k([f]_{U_i}) = j_n f(j_n"\kappa^{+i})$. Furthermore $j = k_i \circ j_i$, where $k_i : M_i \to M$ is given by $k_i([f]_{U_i}) = jf(j"\kappa^{+i})$ and the same when i is replaced by n. For each $\alpha < \kappa^{+\omega+1}$, $k_i([x \mapsto f_\alpha(\kappa_x)]_{U_i}) = [x \mapsto jf_\alpha(j(\kappa) \cap x)]_{U_i}(j"\kappa^{+i}) = jf_\alpha(j(\kappa) \cap j"\kappa^{+i}) = jf_\alpha(\kappa) = \alpha$. So $\mathrm{crit}(k_i) \geqslant \kappa^{+\omega+1}$. By the same argument, $\mathrm{crit}(k_n) \geqslant \kappa^{+\omega+1}$. We then must also have $\mathrm{crit}(k) \geqslant \kappa^{+\omega+1}$. So, if $(\kappa^{+n})^{M_i} \neq \kappa^{+n}$, then letting $\gamma := (\kappa^{+n})^{M_i}$, $M_i \models \gamma$ is a cardinal. But $M_n \models \gamma$ is not a cardinal because M_n computes cardinals $\leqslant \kappa^{+n}$ correctly. Let $h : \gamma \to \kappa^{+m}$ be a bijection in M_n for some $\kappa^{+m} < \gamma$. Then $k^{-1}(h)$ is a bijection from γ to κ^{+m} in M_i, which is a contradiction.

Since the limit of a sequence of cardinals is a cardinal, $\mathbb{D}^*$ preserves $\kappa^{+\omega}$ as well.

For each n, $\mathrm{crit}(j_n) = \kappa$ and $\mathbb{C}_n$ has size $|j_n(\kappa)|^{V_2} \leqslant \kappa^{|P_\kappa(\kappa^{+n})|} = \kappa^{\kappa^{+n}} \leqslant (2^\kappa)^{\kappa^{+n}} = 2^{\kappa^{+n}} = \kappa^{+\omega+1}$. So any antichain has size $< \kappa^{+\omega+2}$, and $\mathbb{D}^*$ preserves cardinals $\geqslant \kappa^{+\omega+2}$. Since $\mathbb{D}^*$ is countably closed, if $\mathbb{D}^*$ collapses $\kappa^{+\omega+1}$, it must collapse it to some cardinal $< \kappa^{+\omega}$. But then $\kappa^{+\omega}$ would also be collapsed, which we know is not the case. So $\mathbb{D}^*$ must preserve $\kappa^{+\omega+1}$. □

Note that since $\mathbb{D}^*$ preserves $\kappa^{+\omega+1}$ and projects to $\mathbb{D}$, $\kappa^{+\omega+1}$ remains cardinal in $V_2[R]$.

Lemma 12. $\mathbb{P}/R = \{p \in \mathbb{P} : \pi(p) \in R\}$ *has the* $\kappa^{+\omega+1}$*-chain condition.*

Proof. In this proof, x_n and c_n are the relevant terms in the generic sequences $\langle x_n : n < \omega\rangle$ and $\langle c_n : n < \omega\rangle$. Towards a contradiction, let $A = \{p_\gamma : \gamma < \kappa^{+\omega+1}\} \subset \mathbb{P}/R$ be an antichain of size $\kappa^{+\omega+1}$. By thinning out A, we may assume every p_γ has the same length $\bar{n}$. For any $\gamma < \kappa^{+\omega+1}$, $A_n^{p_\gamma}$ has measure 1. Also, $\{p \in \mathbb{P}/R : \forall$ large enough $n(A_n^p \subset A_n^{p_\gamma} \wedge \forall x \in A_n^p(C_n^p(x) \leqslant C_n^{p_\gamma}(x)))\}$ is dense. So we may find $p \in G_R$ such that for all large enough n, $A_n^p \subset A_n^{p_\gamma}$ and $C_n^p(x) \leqslant C_n^{p_\gamma}(x)$ for every $x \in A_n^p$ and in particular for $x = x_n$. Let $n_\gamma > \bar{n}$ be such that $x_n \in A_n^{p_\gamma}$ and $c_n \supset C_n^{p_\gamma}(x_n)$ for all $n \geqslant n_\gamma$. Let $I \subset \kappa^{+\omega+1}$ be an unbounded set in $V_2[R][G_R]$ such that $\forall \gamma \in I$, $n_\gamma = N$ for some N.

For $\gamma \in I$, extend each p_γ to q_γ with $\mathrm{length}(q_\gamma) = N$ while keeping $q_\gamma|[N, \omega) = p_\gamma|[N, \omega)$. Since $|I| = (\kappa^{+\omega+1})^{V_2}$ while there are only $(\kappa^{+\omega})^{V_2}$-many stems, and $V_2[R][G_R] \vDash |(\kappa^{+\omega+1})^{V_2}| = \kappa^+ \wedge |(\kappa^{+\omega})^{V_2}| = \kappa$, there are distinct $\gamma_1, \gamma_2 \in I$ such that $q_1 := q_{\gamma_1}$ and $q_2 := q_{\gamma_2}$ have the same stem. Let $h = \mathrm{stem}(q_1) = \mathrm{stem}(q_2)$.

Working in $V_2[R]$, let $D_n = \{y : C_n^{p_{\gamma_1}}(y) \not\perp C_n^{p_{\gamma_2}}(y)\}$. Then $x_n \in D_n$, as witnessed by c_n. By genericity, D_n must be measure 1 for large enough n. Let $N' \geqslant N$ be such that $D_n \in U_n$ for $n \geqslant N'$. For $n \geqslant N'$, let C_n be a common extension of $C_n^{q_1}, C_n^{q_2}$ defined on $A_n^{q_1} \cap A_n^{q_2} \cap D_n$. Then $q := h^\frown\langle x_n, C_n^{q_1}(x_n) \cup C_n^{q_2}(x_n) : N \leqslant n < N'\rangle^\frown\langle A_n^{q_1} \cap A_n^{q_2} \cap D_n, C_n : N' \leqslant n < \omega\rangle$ is a common extension. So q_{γ_1} and q_{γ_2} are compatible, which is a contradiction. $\square$

We now show the first two properties of V_3 promised in Theorem 1.

Lemma 13. *$V_3 \vDash \aleph_{\omega^2}$ is strong limit, $GCH_{\aleph_{\omega^2}}$.*

Proof. We start with $V_2 \vDash \kappa$ strong limit, $2^{\kappa^{+\omega}} = \kappa^{+\omega+1}$. By Lemmas 8 and 9, $\mathbb{D}$ adds no new subsets of $\kappa^{+\omega}$ while preserving all cardinals $\leqslant \kappa^{+\omega+1}$. So $V_2[R] \vDash \kappa$ strong limit, $2^{\kappa^{+\omega}} = \kappa^{+\omega+1}$.

To show $\aleph_{\omega^2}$ is strong limit, since $\langle \kappa_n : n < \omega\rangle$ is cofinal in $\aleph_{\omega^2}$, it suffices to show that $V_3 \vDash 2^{\kappa_n} < \aleph_{\omega^2}$. $\mathbb{P}/R$ adds generics for $\mathrm{Col}(\kappa_n^{+\omega+2}, < \kappa_{n+1})$; so by standard results $V_3 \vDash 2^{\kappa_n^{+\omega+2}} = \kappa_{n+1}$. Hence $V_3 \vDash 2^{\kappa_n} \leqslant \kappa_{n+1} < \aleph_{\omega^2}$.

To show $GCH_{\aleph_{\omega^2}}$, we use the following standard result on powersets in generic extensions.

Fact 14. *Let $\mathbb{P} \in V$ have size λ^+ and the λ^+-chain condition and G be generic for $\mathbb{P}$ over V. Suppose λ is collapsed to κ while $(\lambda^+)^V = (\kappa^+)^{V[G]}$. Then $V[G] \vDash 2^\kappa \leqslant ((\lambda^+)^\lambda)^V$.*

For conditions $p = \langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}, A_n, C_n, ...\rangle \in \mathbb{P}/R$, there are $\leqslant \kappa$-many values for d, $\leqslant \kappa^{+i}$-many values for x_i, $\leqslant \kappa$-many values for c_i, $\leqslant \kappa^{+\omega+1}$-many values for A_i and $\leqslant \kappa^{+\omega+1}$-many values for C_i. So $\mathbb{P}/R$ has size $\kappa^{+\omega+1}$. By the fact just proved, $V_3 \vDash 2^\kappa \leqslant ((\kappa^{+\omega+1})^{\kappa^{+\omega}})^{V_2[R]} = (\kappa^{+\omega+1})^{V_2[R]}$. Hence $V_3 \vDash 2^{\aleph_{\omega^2}} = \aleph_{\omega^2+1}$. $\square$

Lemma 15. *S is stationary and reflects stationarily often in V_3*

Proof. First note that $\mathbb{D}$ preserves stationarity of S because it is $\kappa^{+\omega} + 1$-strategically closed, and by a standard result, $\omega + 1$-strategically closed forcings preserve stationarity of sets with points of countable cofinality.

By Lemma 12, because of the chain condition, passing from $V_2^{\mathbb{D}}$ to V_3 preserves stationary subsets of $\kappa^{+\omega+1}$. So S remains stationary in V_3.

Let $T \in V_2$ be the set of reflection points of S. We first show that T is stationary in V_2. Suppose not and let $C \subset \kappa^{+\omega+1}$ be a club in V_2 disjoint from T. Then S does not reflect at any point of C. By elementarity, $j(S)$ does not reflect at any point of $j(C)$. Now $C^* = \{x \in P_\kappa(\kappa^{+\omega+1}) : \sup x \in C\}$ is a club, hence $C^* \in U$ and $\forall_U x \ \sup x \in C$. Since $\rho = \sup j"\kappa^{+\omega+1} = [x \mapsto \sup x]_U$, by Los' Theorem we have $\rho \in j(C)$. So $j(S) \cap \rho$ cannot be stationary. On the other hand, let $B \subset \rho$ be a club. Then $B' = \{\alpha < \kappa : j(\alpha) \in B\}$ is a $< \kappa$-club. Let $\bar{B}' = B' \cup \{\alpha < \kappa : \alpha \text{ is a limit point of } B'\}$. This is a club, so let $\delta \in \bar{B}' \cap S$. Since $S \subset \mathrm{cof}(\omega)$, $\delta \in B' \cap S$. Then $j(\delta) \in B \cap j(S)$, showing that $j(S) \cap \rho$ is stationary. This is a contradiction. Therefore T must be stationary in V_2.

Note that κ remains supercompact in $V_2^{\mathbb{D}^*}$ because $\mathrm{Add}(\kappa, \kappa^{+\omega+1}) * \mathbb{D}^*$ is κ-directed closed and S remains stationary in $V_2^{\mathbb{D}^*}$ because $\mathbb{D}^*$ is $\omega + 1$ strategically closed and $S \subset \mathrm{cof}(\omega)$. Since the previous paragraph only used the supercompactness of κ and the fact that S is stationary in V_2, $T^* \doteq \{\alpha : S \cap \alpha \text{ stationary in } V_2^{\mathbb{D}^*}\}$ must also be stationary. But reflection at α is downwards absolute, hence $T^* \subset T$. So T must be stationary in $V_2[R]$ as well. Since $\mathbb{P}/R$ has the $\kappa^{+\omega+1}$-chain condition, T is stationary in V_3.

It remains to show that stationarily many points of T remain reflection points of S in V_3. It suffices to do this for all reflection points of any prescribed uncountable cofinality; but we will do it for many such cofinalties. Passing from V_2 to $V_2[R]$ preserves all reflection points because $\mathbb{D}$ is $\kappa^{+\omega}+1$-strategically closed, and so no new bounded subsets of κ are added. Let α be a reflection point in $V_2[R]$ and $\tau = \mathrm{cf}(\alpha)$.

Lemma 16. *If $\tau \in (\kappa_n, \kappa_n^{+\omega+2})$ for some n, τ is regular in V_3 and $V_3 \models A \subset ON, ot(A) = \tau$, then $\exists B \in V_2$ such that $B \subset A$ is unbounded. In particular, if $cf^{V_3}(\tau) \in (\kappa_n, \kappa_n^{+\omega+2})$, then $cf^{V_3}(\tau) = cf^{V_2}(\tau)$.*

Proof. Let $p \in \mathbb{P}$ be such that $p \Vdash \dot{h} : \tau \to A$ is increasing and cofinal. Without loss of generality length$(p) > n$. For each $\beta < \tau$, let $D_\beta = \{q : \exists \gamma \in A \ q \Vdash \dot{h}(\beta) = \gamma\}$. By Lemma 6, we can find $p_\beta \leqslant^* p$ and n_β such that $\forall q \leqslant p_\beta$ with length$(q) = n_\beta$, $q \in D_\beta$. Using the notation $p_\beta = \langle d^\beta, x_0^\beta, c_0^\beta, ..., x_{m-1}^\beta, c_{m-1}^\beta, A_m^\beta, C_m^\beta, ...\rangle$, we may assume we have constructed the p_β inductively so that $A_i^\beta \subset A_i^\gamma$ for $\beta > \gamma$ and $\langle [C_i^\beta]_{U_i} : i < \tau\rangle$ is decreasing; we can do this because $\mathbb{C}_i$ is $\kappa_{M_i}^{+\omega+2}$-closed, which is larger than τ. Our goal will be to find an unbounded $I \subset \tau$ and $p' \in \mathbb{P}$ a lower bound for $\langle p_\beta : \beta \in I\rangle$. We will do this coordinate by coordinate. This is already the case with the x_i coordinates, so we need only focus on the d coordinate and the c_i coordinates.

Since $c_i \in \mathrm{Col}(\kappa_i^{+\omega+2}, < \kappa_{i+1})$, which is $\kappa_i^{+\omega+2}$ closed, whenever $\tau < \kappa_i^{+\omega+2}$, i.e. $n \leqslant i$, we can take the c_i^β decreasing and then find a lower bound. Now consider those $i < n$. For each $(d, c_0, ...c_{n-1}) \in \mathrm{Col}(\omega_1, < \kappa_0) \times \prod_{i<n} \mathrm{Col}(\kappa_i^{+\omega+2}, < \kappa_{i+1})$, let $A_{(d,c_0,...,c_{n-1})} = \{\beta < \tau : d^\beta = d, \forall i(c_i^\beta = c_i)\}$. This is a partition of τ into $|\mathrm{Col}(\omega_1, < \kappa_0) \times \prod_{i<n} \mathrm{Col}(\kappa_i^{+\omega+2}, < \kappa_{x_{i+1}})| = \kappa_n < \tau$ pieces. Since τ is regular, there is some $(d, c_0, ..., c_{n-1})$ such that $I = A_{(d,c_0,...,c_{n-1})}$ is unbounded. $\langle d^\beta : \beta \in I\rangle$ and $\langle c_i^\beta : \beta \in I\rangle$ are constant sequences, so a lower bound is just those constants.

Now let p' be a lower bound for $\langle p_\beta : \beta \in I\rangle$ using again the closure of the $\mathbb{C}_i$. By further shrinking I, we may assume that $n_\beta = k$ for some constant k on I. Let $q \leqslant p'$ with length$(q) = k$. Then q decides the value of $\dot{h}(\beta)$ for every $\beta \in I$. Letting $B = \{\gamma : \exists\beta \in I(q \Vdash \dot{h}(\beta) = \gamma)\}$, we get the desired unbounded subset in V_2. $\square$

Suppose $\tau = \mathrm{cf}(\alpha)$ is as in the lemma just proved. If α is no longer a reflection point in V_3, let $\langle \alpha_i : i < \tau\rangle$ enumerate a club with suprenum α in $V_2[R]$ and $C \subset \alpha$ be a club in V_3 such that $S \cap \alpha \cap C = \varnothing$. Let $A = \{i < \tau : \alpha_i \in C\}$. Then A is unbounded; so by the claim there is unbounded $A' \subset A$ in $V_2 \subset V_2[R]$. But then letting C' be the closure of $\{\alpha_i : i \in A'\}$ gives a club in $V_2[R]$ disjoint from $S \cap \alpha$, which is a contradiction. So α remains a reflection point in V_3. $\square$

4. A Very Good Scale and a Bad Scale

We will now show that there is a very good scale and a bad scale in V_3, as promised in Theorem 1. Throughout this section, we will write $f \leqslant^* g$ for scales f and g to mean that g eventually dominates f.

Motivated by arguments in [1], we first prove a Bounding Lemma.

Lemma 17. ***Bounding Lemma***
Let $\langle \eta(n) : n < \omega\rangle$ be a sequence of ordinals such that $n \leqslant \eta(n) \leqslant \omega$. Then for any $t \in \prod_n \kappa_n^{+\eta(n)+1}$, there is a sequence $\langle B_n : n < \omega\rangle \in V_2$ with B_n an ordinal-valued function on $P_\kappa(\kappa^{+n})$ such that on a U_n-measure one set of x, $B_n(x) < \kappa_x^{+\eta(n)+1}$ and for all large enough n, $t(n) < B_n(x_n)$.

Proof. Let $p \Vdash \dot{t} \in \prod_n \dot{\kappa_n}^{+\eta(n)+1}$. For $n \geqslant m :=$ length(p) and conditions q of length $> n$, write $q||t(n)$ to mean $q \Vdash \dot{t}(n) = \beta$ for some $\beta < \kappa_{x_n^q}$. Our goal will be to define B_n and a condition p^n with stem$(p^n) =$ stem(p) such that $p^n \Vdash \dot{t}(n) < B_n(\dot{x_n})$. We will first define for each $x \in P_\kappa(\kappa^{+n})$ a suitable upper part $u_x = \langle x, c_x\rangle^\frown\langle A^{n+1}_x, C^{n+1}_x, ...\rangle$. We may assume $x^p_{m-1} \prec x$ since this happens for U_m-almost all x whenever $n \geqslant m$.

Fix x. Let S_x be the set of all stems h of length n extending stem(p) such that $h^\frown\langle x, C^p_n(x)\rangle$ is also a stem. Since for every $k \leqslant n$ and $y \in P_\kappa(\kappa^{+k})$, $|\kappa^{+k} \cap y| = \kappa^{+k}_y$, then for U_n-almost all x we have $|\{y : y \prec x\}| \leqslant |P_{\kappa_x}(\kappa^{+n} \cap x)| \leqslant |\kappa^{+n} \cap x|^{<\kappa_x} = (\kappa^{+n}_x)^{<\kappa_x} = \kappa^{+n}_x$. Then $|S_x| \leqslant \kappa^{+n}_x$. So let $\langle\langle h_\gamma, \beta_\gamma\rangle : \gamma < \kappa^{+\eta(n)+1}_x\rangle$ be an enumeration of $\{\langle h, \beta\rangle : h \in S_x, \beta < \kappa^{+\eta(n)+1}_x\}$. Applying the Prikry Property to the condition $h_\gamma{}^\frown\langle x, C^p_n(x)\rangle^\frown\langle A^p_{n+1}, C^p_{n+1}, ...\rangle$ and the sentence $\dot{t}(n) = \beta_\gamma$, we get a condition $p_\gamma = h'_\gamma{}^\frown\langle x, c^{p_\gamma}_n\rangle^\frown\langle A^{p_\gamma}_{n+1}, C^{p_\gamma}_{n+1}, ...\rangle$ with $h'_\gamma \leqslant^* h_\gamma$, $c^{p_\gamma}_n \leqslant C^p_n(x)$ and $p_\gamma || \dot{t}(n) = \beta_\gamma$. Since $\mathrm{Col}(\kappa^{+\omega+2}_x, <\kappa)$ is $\kappa^{+\omega+2}_x$-closed, choose the p_γ inductively so that $\langle c^{p_\gamma}_n : \gamma < \kappa^{+\eta(n)+1}_x\rangle$ is decreasing and let c_x be a lower bound. Inductively, define for $k \geqslant n+1$, $A^x_k = \bigcap_\gamma A^{p_\gamma}_k$. Since $\mathrm{Col}(\kappa^{+\omega+2}_y, <\kappa)$ is $\kappa^{+\omega+2}_y$-closed, we can take the $\langle C^{p_\gamma}_i(y) : \gamma < \kappa^{+\eta(n)+1}_x\rangle$ to be decreasing for each $y \in A^x_k$. Define $C^x_k = \bigcup_\gamma C^{p_\gamma}_k$ and $u_x = \langle x, c_x\rangle^\frown\langle A^x_{n+1}, C^x_{n+1}, ...\rangle$.

With x now allowed to vary, let $p^n = p|n^\frown\langle A^{p^n}_n, C^{p^n}_n, ...\rangle$, where

- $p|n = \langle d^p_0, x^p_0, c^p_0, ..., x^p_{m-1}, c^p_{m-1}, A^p_m, C^p_m, ..., A^p_{n-1}, C^p_{n-1}\rangle$
- $A^{p^n}_n = A^p_n$
- $C^{p^n}_n(x) = c_x$
- $A^{p^n}_k = \Delta_{x \in P_\kappa(\kappa^{+n})} A^k_x$ for $k > n$
- $C^{p^n}_k(y) = \bigcup_{x \prec y} C^x_k(y)$ on a measure 1 subset of $A^{p^n}_k$

Define $B_n(x) = \sup\{\beta : \exists q(\mathrm{length}(q) = n+1, q|[n, \omega) = u_x, q \Vdash \dot{t}(n) = \beta)\} + 1$. There are $\leqslant \kappa^{+n}_x$ choices for q in the definition of $B_n(x)$. Since the β corresponding to each q is below $\kappa^{+\eta(n)+1}_x$, $B_n(x) < \kappa^{+\eta(n)+1}_x$. It remains to show that $p^n \Vdash \dot{t}(n) < B_n(\dot{x}_n)$ for all large enough n. Let $q \leqslant p^n$ have length $n+1$. Then q is of the form $h^\frown\langle x, c\rangle^\frown q|[n+1, \omega)$. By the construction above, there is $p_\gamma \leqslant^* h^\frown\langle x, c^{h_\gamma}_n\rangle^\frown\langle A^p_{n+1}, C^p_{n+1}, ...\rangle$ such that $p_\gamma \Vdash \dot{t}(n) = \beta$ for some β. Since $p_\gamma|n^\frown u_x \leqslant p_\gamma$, $p_\gamma|n^\frown u_x \Vdash \dot{t}(n) = \beta$. It follows that $\beta < B_n(x)$.

Let $h' = p_\gamma|n \leqslant^* h$ and $q' = h'^\frown\langle x, c\rangle^\frown q|[n+1, \omega)$. Then $q' \leqslant q$ because $c \leqslant c_x$ and $\mathrm{stem}(q') \leqslant^* \mathrm{stem}(p_\gamma)$. We need to show $q'|[n+1, \omega) \leqslant p_\gamma|[n+1, \omega)$ as well. Let $k \geqslant n+1$. Then $A^{q'}_k \subset A^{p^n}_k = \{z : z \in \bigcap_{y \prec z} A^k_y\}$. If $z \in A^{q'}_k$, then $x \prec z$, so $z \in A^k_x \subset A^{p_\gamma}_k$. Hence $A^{q'}_k \subset A^{p_\gamma}_k$. If $z \in A^{q'}_k$, then $z \in A^{p_\gamma}_k$ and $C^{q'}_k(z) \leqslant C^{p^n}_k(z) = \bigcup_{y \prec z} C^y_k(z)$. But $x \prec z$, so $C^{q'}_k(z) \leqslant C^x_k(z) \leqslant C^{p_\gamma}_k(z)$, which shows $C^{q'}_k \subset C^{p_\gamma}_k$. Therefore, $q' \leqslant p_\gamma$, which gives $q' \Vdash \dot{t}(n) = \beta$ and $q' \Vdash \dot{t}(n) < B_n(\dot{x}_n)$. By density of the q' below p^n, $p^n \Vdash \dot{t}(n) < B_n(\dot{x}_n)$.

Finally, observe that we could have constructed the p^n inductively so that $\langle p^n : n < \omega\rangle$ is decreasing. Assume we have done so and let p^* be a lower bound. Then $p^* \Vdash \dot{t}(n) < B_n(\dot{x}_n)$ for all $n \geqslant m$. Since the argument works densely below p, we have the Bounding Lemma. $\square$

Theorem 18. *There is a very good scale $\langle t_\alpha : \alpha < \kappa^+\rangle \in V_3$ for κ.*

Proof. Recall that $\kappa_n := \kappa \cap x_n$ and in Section 2, we found $f_\alpha : \kappa \to \kappa$ for $\alpha < \kappa^{+\omega+1}$ such that $jf_\alpha(\kappa) = \alpha$. For each $\alpha < \kappa^+$, define in V_3, $t_\alpha(n) = f_\alpha(\kappa_n)$ if $f_\alpha(\kappa_n) < \kappa_n^{+\omega+1}$ and 0 otherwise. The proof that this is a very good scale is as in [4]. $\square$

Theorem 19. *There is a bad scale $\langle g_\beta : \beta < \kappa^+\rangle \in V_3$ for κ.*

Proof. Shelah [7] showed every scale above a supercompact cardinal is bad. We include the proof of the following more specific result for completeness.

Lemma 20. *Let $V \vDash \kappa$ supercompact and $\langle h_\beta : \beta < \kappa^{+\omega+1}\rangle$ be a scale in $\prod_n \kappa^{+n+1}$. Then there is inaccessible $\delta < \kappa$ such that there are stationarily many bad points of cofinality $\delta^{+\omega+1}$.*

Proof. Let us write $\vec{h}$ for $\langle h_\beta : \beta < \kappa^{+\omega+1}\rangle$. Towards a contradiction, assume no such δ exists. Then for every inaccessible $\delta < \kappa$, there is a club $C_\delta \subset \kappa^{+\omega+1}$ such that every $\beta \in C_\delta$ with $\mathrm{cf}(\beta) = \delta^{+\omega+1}$ is a good point for $\vec{h}$. Let $C = \bigcap_\delta C_\delta$. This is still a club.

Suspending the previous definitions of V, M and U for the rest of this lemma, let $j : V \to M$ be a $\kappa^{+\omega+1}$-supercompact embedding with corresponding normal measure U and $\rho = \sup j``(\kappa^{+\omega+1})$. Then $\rho = [x \mapsto \sup x]_U$ and $j(C) = [x \mapsto C]_U$. Since $\sup x \in C$ for U-almost all (in fact for club many) x, $M \vDash \rho \in j(C)$. Now $V \vDash \forall\beta\forall\delta$ inacc $(\beta \in C \wedge \mathrm{cf}(\beta) = \delta^{+\omega+1} \Rightarrow$ β is a good point for $\vec{h}$. So $M \vDash \forall\beta\forall\delta$ inacc $(\beta \in j(C) \wedge \mathrm{cf}(\beta) = \delta^{+\omega+1} \Rightarrow$ β is a good point for $j(\vec{h})$. Since κ is inaccessible in M and $\mathrm{cf}(\rho) = \kappa^{+\omega+1}$, ρ is a good point for $j(\vec{h})$.

Working in M, define $f(n) = \sup(j``\kappa^{+n+1})$. We will show $f$ is an eub for $\langle j(h)_\beta : \beta < \rho\rangle$. Given $\beta < \rho$, let $\gamma < \kappa^{+\omega+1}$ be such that $j(\gamma) > \beta$. Then $j(h)_\beta <^* j(h)_{j(\gamma)} = j(h_\gamma)$. Since $h_\gamma(n) < \kappa^{+n+1}$, $j(h_\gamma)(n) < \sup j``(\kappa^{+n+1}) = f(n)$, we have $j(h)_\beta <^* f$. So f is an upper bound. Now let $h <^* f$, i.e. $h(n) < \sup(j``\kappa^{+n+1})$ for large enough n. Then for n large enough, there are $\gamma_n < \kappa^{+n+1}$ such that $h(n) < j(\gamma_n)$. Let $\bar{h}(n) = \gamma_n$. Since $\vec{h}$ is a scale, we can find β so that $h_\beta >^* \bar{h}$. Then $j(h_\beta) = j(h)_{j(\beta)} >^* j(\bar{h}) >^* h$. Hence f is exact. But $\mathrm{cf}(f(n)) = \kappa^{+n+1}$, which is strictly increasing. Therefore, ρ is a bad point for $j(\vec{h})$. This is a contradiction. $\square$

Working in V_2, since κ is supercompact, there is a bad scale $\langle G_\beta : \beta < \kappa^{+\omega+1}\rangle$ on $\prod_n \kappa^{+n+1}$ with stationarily many bad points. For each n and each $\eta < \kappa^{+n+1}$, let F_n^η be a representative for η in $Ult(V_2, U_n)$, i.e. $[F_n^\eta]_{U_n} = \eta$

and for U_n-almost all x, $F_n^\eta(x) < \kappa^{+n+1}$. Note that $Ult(V_2, U_n)$ computes κ^{+n+1} correctly because it is closed under κ^{+n} sequences.

In V_3, define $\langle g_\beta : \beta < \kappa^+\rangle$ by $g_\beta(n) = F_n^{G_\beta(n)}(x_n)$, recalling that $\langle x_n : n < \omega\rangle$ is the Prikry sequence added by $\mathbb{P}$. Note that this is well-defined for n large enough by Lemma 8 because $\langle \mathrm{dom}(F_n^{G_\beta(n)}) : n < \omega\rangle$ is a sequence of measure 1 sets.

Subclaim 21. *If $\alpha < \beta$, then $g_\alpha(n) < g_\beta(n)$ for large enough n.*

Proof. Since $\langle G_\beta : \beta < \kappa^{+\omega+1}\rangle$ is a scale, $G_\alpha(n) < G_\beta(n)$ for large enough n. So for large enough n, $F_n^{G_\alpha(n)}(x) < F_n^{G_\beta(n)}(x)$ for U_n-almost all x. It follows that $g_\alpha(n) = F_n^{G_\alpha(n)}(x_n) < F_n^{G_\beta(n)}(x_n) = g_\beta(n)$ for large enough n. □

Subclaim 22. *$\langle g_\beta : \beta < \kappa^+\rangle$ is a scale in $\prod_n \kappa_n^{+n+1}$*

Proof. We need to show that for any $h \in \prod_n \kappa_n^{+n+1}$ in V_3, there is β such that $h(n) < g_\beta(n)$ for large enough n. For this, we use the Bounding Lemma with $\eta(n) = n$ to get $\langle B_n : n \in \omega\rangle \in V_2$ such that $[B_n]_{U_n} < \kappa^{+n+1}$ and for large enough n, $h(n) < B_n(x_n)$. Since $\langle G_\beta : \beta < \kappa^{+\omega+1}\rangle$ is cofinal in $\prod_n \kappa^{+n+1}$, let β be such that $[B_n]_{U_n} < G_\beta(n)$ for large enough n. Then $[B_n]_{U_n} < [F_n^{G_\beta(n)}]_{U_n}$, i.e. $B_n(x) < F_n^{G_\beta(n)}(x)$ for U_n-almost all x. It follows that $h(n) < B_n(x_n) < F_n^{G_\beta(n)}(x_n) = g_\beta(n)$ for large enough n. □

Lemma 23. *Let α be a bad point of $\langle G_\beta : \beta < \kappa^{+\omega+1}\rangle$ with cf$(\alpha) \in (\kappa_n, \kappa_n^{+\omega+2})$ for some n. Then α is a bad point of $\langle g_\beta : \beta < \kappa^+\rangle$.*

Proof. We will show the contrapositive. By Lemma 16, we know that for any such α, $\mathrm{cf}^{V_3}(\alpha) = \mathrm{cf}^{V_2}(\alpha)$ so that in particular, $\omega < \mathrm{cf}^{V_2}(\alpha) < \kappa$. Let unbounded $A \subset \alpha$ and m witness that α is a good point of $\langle g_\beta : \beta < \kappa^+\rangle$, i.e. $\langle g_\beta(n) : \beta \in A\rangle$ is strictly increasing for any fixed $n \geqslant m$. By Lemma 16, A has an unbounded subset in V_2; so we may assume that $A \in V_2$.

Let $p = \langle x_0, c_0, ..., x_{k-1}, c_{k-1}, A_k, C_k, ...\rangle$ be such that
$p \Vdash \forall n \geqslant m \forall \beta < \beta' \in A(\dot{g_\beta}(n) < \dot{g_{\beta'}}(n))$. Without loss of generality, assume $k \geqslant m$. Then in particular, $p \Vdash \forall \beta < \beta' \in A(F_k^{G_\beta(k)}(\dot{x}_k) < F_k^{G_{\beta'}(k)}(\dot{x_k}))$.

Let $B = A_k \cap \{x : \forall \beta < \beta' \in A(F_k^{G_\beta(k)}(x) < F_k^{G_{\beta'}(k)}(x))\}$. We claim that $B \in U_k$. If not, $B' = A_k \cap \{x : \neg\forall \beta < \beta' \in A(F_k^{G_\beta(k)}(x) < F_k^{G_{\beta'}(k)}(x))\}$. We can then take $q \leqslant p$ of the form $q = \langle x_0, c_0, ..., x_{k-1}, c_{k-1}, B', C_k^q, ...\rangle$. Then $q \Vdash \dot{x}_k \in B'$, which is a contradiction because $p \Vdash \dot{x}_k \notin B'$.

We now have $\forall \beta < \beta' \in A(F_k^{G_\beta(k)}(x) < F_k^{G_{\beta'}(k)}(x))$ for U_k-almost all x.

But $[F_k^\beta]_{U_k} = \beta$ So $\forall \beta < \beta' \in A(G_\beta(k) < G_{\beta'}(k))$. Since this argument works for any $k \geqslant m$, α is a good point of $\langle G_\beta : \beta < \kappa^{+\omega+1}\rangle$. $\square$

Let $B \in V_2$ be the set of bad points of $\langle G_\beta : \beta < \kappa^{+\omega+1}\rangle$. This is stationary. The goal is to show that if $B' \in V_3$ is the set of bad points of $\langle g_\beta : \beta < \kappa^+\rangle$, then this is stationary as well.

Recall from Lemma 11 that $\mathbb{D}^*$ preserves $\kappa^{+\omega+1}$ and every κ^{+n+1}. $\langle G_\beta : \beta < \kappa^{+\omega+1}\rangle$ remains a scale in $V_2^{\mathbb{D}^*}$ because being increasing under the eventual domination ordering is absolute, and it remains cofinal because $\mathbb{D}^*$ is κ-closed hence adds no new ω-sequences. Let $\bar{B}$ be its set of bad points. Since κ remains supercompact, by Lemma 20 there is inaccessible δ such that $\bar{B} \cap \mathrm{cof}(\delta^{+\omega+1})$ is stationary. But $\bar{B} \subset B$ because being a bad point is downwards absolute. So $B \cap \mathrm{cof}(\delta^{+\omega+1}) \in V_2^{\mathbb{D}^*}$ is stationary.

Since stationarity is downwards absolute, $B \cap \mathrm{cof}(\delta^{+\omega+1})$ is stationary in $V_2[R]$. Since $\mathbb{P}/R$ has the $\kappa^{+\omega+1}$-chain condition, $B \cap \mathrm{cof}(\delta^{+\omega+1})$ remains stationary in V_3. Assume we have forced below a condition making $\kappa_0 = \delta$. Then $B' \supset B \cap \mathrm{cof}(\delta^{+\omega+1})$ by Lemma 23. So B' is stationary. $\square$

5. The Failure of Diamond

We now prove that $\diamond_S$ fails in V_3, as promised in Theorem 1.

Theorem 24. $V_3 \models \neg\diamond_S$

The following lemma is a standard result due to Kunen.

Lemma 25. *Let τ be a cardinal, $S \subset \tau^+$ be stationary and $V \models \neg\diamond_S$. Then for any generic G of a τ^+-cc forcing $\mathbb{P}$, $V[G] \models \neg\diamond_S$.*

It follows that passing from V_1 to V_2^-, V_2^- to V_2 and $V_2[R]$ to V_3 all preserve $\neg\diamond_S$. It remains to show that passing from V_2 to $V_2[R]$ preserves $\neg\diamond_S$.

Lemma 26. *$\bar{M}$ does not have a $\diamond_S$ sequence.*

Proof. Suppose $\bar{M} \models \langle A_\alpha : \alpha \in S\rangle$ is a $\diamond_S$ sequence. Then $\langle A_\alpha : \alpha \in S\rangle \in V_2$. Since $\bar{M}$ is closed under $\kappa^{+\omega+2}$-sequences, given $A \subset \kappa^{+\omega+1}$ in V_2, $A \in \bar{M}$. So $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary in $\bar{M}$. But every club subset of $\kappa^{+\omega+1}$ in V_2 is in $\bar{M}$. So $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary in V_2, giving us a $\diamond_S$ sequence in V_2, a contradiction. $\square$

Lemma 27. *Let $\mathbb{Q}$ be a $\kappa^{+\omega+2}$-closed forcing over $\bar{M}$. Then $\mathbb{Q}$ does not add $\diamond_S$ with respect to V_2 sets, i.e. there is no $\langle S_\alpha : \alpha \in S\rangle$ such that for every $A \subset \kappa^{+\omega+1}$ in V_2, $\{\alpha : A \cap \alpha = S_\alpha\}$ is stationary.*

Proof. Note that subsets of $\kappa^{+\omega+1}$ in $\bar{M}$ are the same as subsets of $\kappa^{+\omega+1}$ in V_2.

Suppose for contradiction $\mathcal{S} = \langle S_\alpha : \alpha \in S\rangle$ is a $\diamond_S$ sequence with respect to V_2 sets in $\bar{M}^{\mathbb{Q}}$ and let $p \in \mathbb{Q}$ be such that $p \Vdash \dot{\mathcal{S}}$ is a $\diamond_S$ sequence with respect to V_2 sets. Since $\mathbb{Q}$ is $\kappa^{+\omega+2}$-distributive, each $S_\alpha \in \bar{M}$ is such that for any $A \in \bar{M}$, $\{\alpha \in S : A \cap \alpha = S_\alpha\}$ is stationary. We will show that $\mathcal{S} \in \bar{M}$. This will be a contradiction because $\bar{M} \models \neg\diamond_S$.

Inductively define $\langle p_\alpha : \alpha \in S\rangle$ decreasing so that $p_\alpha \Vdash \dot{\mathcal{S}}(\alpha) = S_\alpha$. Since $|S| < \kappa^{+\omega+2}$, there is a lower bound q. Then $\mathcal{S} = \{(\alpha, A) : \alpha \in S, q \Vdash \dot{\mathcal{S}}(\alpha) = A\} \in \bar{M}$. □

Let $\bar{\mathbb{C}} = [x \mapsto \mathrm{Col}(\kappa_x^{+\omega+2}, < \kappa)]_{\bar{U}} = \mathrm{Col}^{\bar{M}}(\kappa^{+\omega+2}, < j_{\bar{U}}(\kappa))$. By Lemma 27, $\bar{M}^{\bar{\mathbb{C}}}$ does not have a $\diamond_S$ sequence. Let $\bar{k}_n : M_n \to \bar{M}$ be given by $\bar{k}_n([f]_{U_n}) = j_{\bar{U}} f(j_{\bar{U}}{}''\kappa^{+n})$. Then $j_{\bar{U}} = \bar{k}_n \circ j_n$. By the argument from Lemma 11, $\mathrm{crit}(\bar{k}_n) > \kappa^{+\omega+1}$. Since $S \subset \kappa^{+\omega+1}$, $\bar{k}_n(S) = S$.

Lemma 28. *$M_n^{\mathbb{C}_n}$ does not have a $\diamond_S$ sequence.*

Proof. Suppose $\langle A_\alpha : \alpha \in S\rangle$ is a $\diamond_S$ sequence in $M_n^{\mathbb{C}_n}$ and $\dot{h}$ is a $\mathbb{C}_n$ name over M_n for the function $h(\alpha) = A_\alpha$. Let $p \in \mathbb{C}_n$ be such that $p \Vdash \dot{h}$ is a $\diamond_S$ sequence. Then $\bar{k}_n(p) \Vdash \bar{k}_n(\dot{h})$ is a $\diamond_S$ sequence. But this is a contradiction because $\bar{M}^{\bar{\mathbb{C}}}$ does not have a $\diamond_S$ sequence, and we could have forced below $\bar{k}_n(p)$. □

Lemma 29. *Let G_n be generic for $\mathbb{C}_n$ over M_n. Then there is $G^{(n)}$ generic for $\bar{\mathbb{C}}$ over $\bar{M}$ and $k_n^* : M_n[G_n] \to \bar{M}[G^{(n)}]$ extending $\bar{k}_n : M_n \to \bar{M}$ such that $k_n^*(\tau_{G_n}) = \bar{k}_n(\tau)_{G^{(n)}}$ for every $\mathbb{C}_n$-name τ.*

Proof. It is enough to find $p \in \bar{\mathbb{C}}$ such that $p \leqslant k_n(q)$ for every $q \in G_n$. This is possible because $|G_n| \leqslant |\mathbb{C}_n| = \kappa^{+\omega+1}$ and $\bar{\mathbb{C}}$ is $\kappa^{+\omega+2}$-closed and because $\mathrm{crit}(\bar{k}_n) > \kappa^{+\omega+1}$. □

From now on, we use $\prod_n G_n$ as our generic for $\mathbb{D}^*$ over V_2 and $H := \prod_n G^{(n)}$ as our generic for $\prod_n \bar{\mathbb{C}}$ over $\bar{M}$.

Lemma 30. *$V_2^{\mathbb{D}^*}$ does not have a $\diamond_S$ sequence with respect to V_2 sets.*

Proof. For any $q \in \prod_n \mathbb{C}_n$, $q = \langle q_n : n < \omega\rangle$ with $q_n \in \mathbb{C}_n$. Let $\bar{q}_n = \bar{k}_n(q_n) \in \bar{\mathbb{C}}$ and $\bar{q} = \langle \bar{q}_n : n < \omega\rangle \in \prod_n \bar{\mathbb{C}}$.

Let $\langle A_\alpha : \alpha \in S\rangle$ be a $\diamond_S$ sequence with respect to V_2 in $V_2^{\mathbb{D}^*}$, $h(\alpha) = A_\alpha$ and suppose $p \in \prod_n \mathbb{C}_n$, $p \models \dot{h}$ is a $\diamond_S$ sequence with respect to V_2 sets. Without loss of generality, assume $p_n \in G_n$. Then $\bar{p} \in H$.

Let $\dot{A}_\alpha^* = \{\langle \check{\beta}, \bar{q}\rangle : q \Vdash \beta \in \dot{A}_\alpha\}$ a $\prod_n \bar{\mathbb{C}}$-name. Then whenever $q \Vdash \beta \in \dot{A}_\alpha$, $\bar{q} \Vdash \beta \in \dot{A}_\alpha^*$. Let $\dot{h}^*$ be a $\prod_n \bar{\mathbb{C}}$-name for the sequence $\alpha \mapsto A_\alpha^*$, where $A_\alpha^* = (\dot{A}_\alpha^*)_H$. We will show that $\bar{M} \models 1_{\prod_n \bar{\mathbb{C}}} \Vdash$ "$\dot{h}^*$ is a $\diamond_S$

sequence with respect to V_2 sets$''$, contradicting Lemma 27.

Let $A \subset \kappa^{+\omega+1}$ be a set in V_2. Then there is stationary $T \subset S$ such that for all $\alpha \in T$ there is $q \leqslant p$ still in $\prod_n G_n$ with $q \Vdash A \cap \alpha = \dot{A}_\alpha$. Given any $\beta \in A \cap \alpha$, strengthening q if necessary, we may assume $q \Vdash \beta \in \dot{A}_\alpha$, hence $\bar{q} \Vdash \beta \in \dot{A}^*_\alpha$. Since $\bar{q} \in H$, $A \cap \alpha \subset A^*_\alpha$. On the other hand suppose $\beta \notin A \cap \alpha$. This is absolute across all models containing A, so we may take $q \leqslant p$ still in $\prod_n G_n$ so that $q \Vdash \beta \notin \dot{A}_\alpha$. We want to show that $\bar{q} \Vdash \beta \notin \dot{A}^*_\alpha$. If not, then there is a generic filter H' containing $\bar{q}$ such that $\beta \in (\dot{A}^*_\alpha)_{H'} = \{\beta : \exists \langle \check{\beta}, \bar{r} \rangle \in \dot{A}^*_\alpha, \bar{r} \in H'\}$, so there is $\bar{r} \in H'$ such that $\langle \check{\beta}, \bar{r} \rangle \in \dot{A}^*_\alpha$. By definition of $\dot{A}^*_\alpha$, $r \Vdash \beta \in \dot{A}_\alpha$. In particular, $q \perp r$. But $\bar{q} \not\perp \bar{r}$ because they belong to the same filter, which means $\forall n(\bar{k_n}(q_n) \not\perp \bar{k_n}(r_n))$. By elementarity, $\forall n(q_n \not\perp r_n)$, so $q \not\perp r$. This is a contradiction. It follows that $\beta \notin A^*_\alpha$.

Therefore, for every $\alpha \in T$, $\bar{M}[H] \models A \cap \alpha = A^*_\alpha$. But T remains stationary in $\bar{M}[H]$ because of the chain condition. $\square$

We are now ready to finish the proof of Theorem 24. If $V_2[R]$ had a $\diamond_S$ sequence, this sequence would exist in $V_2^{\mathbb{D}^*}$ and guess every $V_2[R]$-subset (and in particular every V_2-subset) of $\kappa^{+\omega+1}$ stationarily often. This contradicts Lemma 30.

We conclude with two problems that remain open:

(1) Can we get Theorem 1 for $\aleph_\omega$? Can we even get $VGS_{\aleph_\omega} + \neg\square^*_{\aleph_\omega}$? The same construction but with only finitely many cardinals between successive terms κ_n and κ_{n+1} in the Prikry sequence would not work because $\kappa^{+\omega+1}$ would no longer be a cardinal in $V_2^{\mathbb{D}}$.

(2) Shelah asked whether it is possible to get $\mathrm{GCH}_\kappa + \neg\diamond_S$, where $S = \kappa^+ \cap \mathrm{cof}(\mathrm{cf}(\kappa))$. The larger S is, the more difficult it is to get $\neg\diamond_S$; so this would be the optimum result in the direction of enlarging S.

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